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# Random Hermitian versus random non-Hermitian operators—unexpected links

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**Abstract**

We present some unexpected links between the spectral properties of ensembles of large Hermitian and non-Hermitian random matrices, within the formalism of free random variables. We address the task of adding non-Hermitian random matrices. To solve this problem, we present a new approach [1] based on a generalization of the notion of the matrix-valued Green's function to the quaternion level, and on an extension of free probability theory, which allows us to introduce the setting of quaternion-valued free probability theory. We use this quaternion construction to solve the problem of a non-Hermitian random matrix of the type $H_1 + \mathrm{i}H_2$, with Hermitian $H_1$, $H_2$ freely independent. Finally, we mention conformal mapping relating spectral properties of Hermitian $H_1 + H_2$ to spectral properties of the non-Hermitian $H_1 + \mathrm{i}H_2$ model.

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## 1. Introduction

Random operators play an important role in many branches of physics [2–5]. Among them, non-Hermitian random operators constitute an important class. For instance, they represent Hamiltonians with dissipation [6], Euclidean Dirac operators in the presence of matter [7–9], PT-symmetric operators in mesoscopic physics [10–13], generators of spectral curves in certain growth processes [14], to mention only some applications. Non-Hermitian operators are challenging also from a mathematical point of view and their distinctive feature, which is that in general their spectra are complex, calls for new methods and insights, since several standard tools used for manipulations of Hermitian operators cannot be extended into the non-Hermitian case.

When one studies complicated problems, it is natural to start from some simplifying assumptions; in the context of operators, these would be small matrices representing operators, so $2 \times 2$, $3 \times 3$, etc. Often such problems can be solved exactly, demonstrating expected generic

  

features of larger operators, but equally often these solutions are not applicable to a description of real, much more complicated systems.

In this paper, we however follow a different path, being of great relevance in various contexts dealing with matrices, namely we choose to work with large matrices. At the same time, to avoid introducing many parameters, we simplify the problem by assuming some particular symmetries of large matrices, and knowledge of their few lowest moments, typically mean and variance, sometimes skewness and curtosis. This approach, called the *large-N limit* (where $N$ represents the size of the matrices in question), is the cornerstone of random matrix theory [2–5], and allows us to obtain many results analytically. Moreover, there exists a natural expansion parameter $1/N$, which makes calculations of finite size corrections meaningful and tractable. From now, we will consider random operators represented by infinite size random matrices drawn from some statistical ensemble, and we will be interested in the distribution of their eigenvalues averaged over the ensemble.

In what follows we discuss only one particular problem, *'can we say anything about the spectral properties of the non-Hermitian random matrix $H_1 + \mathrm{i}H_2$ when we know individual spectral properties of the Hermitian random matrices $H_1$ and $H_2$, which are assumed to be independent?'* Since the averaged spectrum of a non-Hermitian large random matrix is complex and in general fills two-dimensional surfaces on the complex plane, whereas the averaged spectrum of a Hermitian random matrix is real and fills intervals on the real line, connection between the two is far from obvious.

To tackle this puzzle we will make use of the solution of an important problem in Hermitian random matrix theory, which asks about the spectral properties of the (Hermitian) sum $H_1 + H_2$ when the spectral properties of the two summands are known. This problem can be naturally addressed within a setting called *free random variables* (FRV) *calculus* [15, 16], which can be viewed as an extension of classical probability calculus into the case of non-commutative random variables, and the *addition algorithm* will turn out to be conceptually a direct generalization of the well-known addition procedure in the commutative case. This passage,

$$H_1, H_2 \quad \longrightarrow \quad H_1 + H_2, \tag{1}$$

will be demonstrated in section 2, where we show how the *real* eigenvalue spectrum of $H_1 + H_2$ can be read from the two respective spectra of $H_1$ and $H_2$ by means of a special function called the *Green's function*, which in this setting is a *complex* function of a *complex* variable.

In section 3 we extend this construction to the non-Hermitian case, i.e. we consider the passage

$$H_1, H_2 \quad \longrightarrow \quad H_1 + \mathrm{i}H_2. \tag{2}$$

The generalization is non-trivial as we again have two real spectra as the input, but in the output we encounter a *complex* spectrum. We show the solution of this problem by introducing quaternion-valued FRV for the non-Hermitian case [1]. In particular, the Green's function becomes a *quaternion* function of a *quaternion* variable; this can be depicted as

$$\begin{array}{ccc} \text{Hermitian} & \longrightarrow & \text{non-Hermitian} \\ \downarrow & & \downarrow \\ \text{real spectrum} & \longrightarrow & \text{complex spectrum} \\ \downarrow & & \downarrow \\ \text{complex Green's function} & \longrightarrow & \text{quaternion Green's function.} \end{array} \tag{3}$$

We exemplify this construction with a pedagogical example.

In section 4 we demonstrate another link [17]: the existence of conformal transformation, allowing us to infer the shape of the support of the complex spectra and the moments of the

spectra for $H_1 + iH_2$ from the singularities of the spectra of $H_1 + H_2$. We also present a pedagogical example.

Section 5 concludes the paper.

## 2. Addition law for Hermitian random matrices

### *2.1. Addition law in classical probability calculus*

Let us start with the well-known setting of classical probability calculus and consider the *addition problem*, i.e. take two real random variables $x_1$ and $x_2$, assume that they are *independent*,

$$\langle x_1 x_2 \rangle = \langle x_1 \rangle \langle x_2 \rangle, \tag{4}$$

and ask about the probability distribution of their sum, $x_1+x_2$. Let us emphasize that the crucial assumption here is the one of independence, which can also be understood as a prescription to calculate the mixed moments $\langle x_1^n x_2^m \rangle$ out of properties of the individual moments:

$$\langle x_1 x_1 x_2 x_2 \rangle = \langle x_1 x_2 x_1 x_2 \rangle = \langle x_1 \rangle^2 \langle x_2 \rangle^2. \tag{5}$$

In the commutative case we can relate the moments of $x_1 + x_2$ to separate $x_1$ and $x_2$ using Newton's formula,

$$\frac{\langle (x_1 + x_2)^n \rangle}{n!} = \sum_{k=0}^{n} \frac{\langle x_1^k \rangle}{k!} \frac{\langle x_2^{n-k} \rangle}{(n-k)!}, \tag{6}$$

which suggests gathering the moments in the following generating function, called the *characteristic function*:

$$g_x(z) \equiv \sum_{n \geqslant 0} \frac{\langle x^n \rangle}{n!} z^n = \langle \mathrm{e}^{zx} \rangle = \int_{\mathbb{R}} \mathrm{d}\lambda \, \rho_x(\lambda) \, \mathrm{e}^{z\lambda}. \tag{7}$$

For $z = \mathrm{i}q$, the characteristic function is the Fourier transform of the probability density function $\rho_x(\lambda)$. Relation (6) results in the following simple behaviour of $g_{a_1+a_2}(z)$:

$$\ln g_{x_1+x_2}(z) = \ln g_{x_1}(z) + \ln g_{x_2}(z), \tag{8}$$

which means that the logarithm of $g_x(z)$ is additive, giving what we can call the *classical addition law*. This can be rewritten in terms of densities $\rho_x(\lambda)$ as

$$\rho_{x_1+x_2}(\lambda') = \int_{\mathbb{R}} \mathrm{d}\lambda \, \rho_{x_1}(\lambda) \rho_{x_2}(\lambda' - \lambda), \tag{9}$$

which means that adding two independent real classical random variables amounts to perform the usual convolution of their measure densities:

$$\rho_{x_1+x_2} = \rho_{x_1} * \rho_{x_2}. \tag{10}$$

Let us note that we have got the addition law straightforwardly in the language of the moments. We can also define *cumulants* as the coefficients in the expansion of $\ln g_x(z)$:

$$\ln g_x(z) = \sum_{n \geqslant 1} c_{x,n} z^n; \tag{11}$$

they thus obey the *linearity condition*,

$$c_{x_1+x_2,n} = c_{x_1,n} + c_{x_2,n}. \tag{12}$$

As an example, we consider the standard Gaussian distribution (with mean zero and unit variance) for both $x_1$ and $x_2$:

$$\rho_{x_{1,2}}(\lambda) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\lambda^2}{2}}; \tag{13}$$

we get the characteristic function from (7) to be

$$g_{x_{1,2}}(z = \mathrm{i}q) = \mathrm{e}^{-q^2}, \tag{14}$$

and the addition law (9) gives

$$\rho_{x_1+x_2}(\lambda) = \frac{1}{\sqrt{2\pi}\sqrt{2}} \mathrm{e}^{-\frac{\lambda^2}{2(\sqrt{2})^2}}, \tag{15}$$

which is again the centred Gaussian distribution, but with variance $\sqrt{2}$.

### *2.2. Hermitian-free random variables calculus*

We would now like to find an analogue of the above construction for the case of *non-commutative random variables*; let us moreover start from the Hermitian case, being a direct extension of the commutative real case. The crucial question which arises is how to define reasonably the notion of independence and at the same time preserve non-commutativity of random variables. The answer is given by Voiculescu's construction of *free independence* [15], or *freeness* for short.

Two Hermitian random matrices $H_1$ and $H_2$ are called *free* if

$$\langle p_1(H_1) r_1(H_2) p_2(H_1) r_2(H_2) \ldots \rangle = 0 \qquad \text{if} \quad \langle p_i(H_1) \rangle = \langle r_j(H_2) \rangle = 0, \tag{16}$$

where $p_i$ and $r_j$ are polynomials. The basic feature of this definition is that consecutive polynomials should depend on different variables. Note that the expectation values of Hermitian random matrices used here are defined as

$$\langle H \rangle \equiv \left\langle \frac{1}{N} \operatorname{Tr} H \right\rangle_{\mathrm{cl}}, \tag{17}$$

with $\langle \ldots \rangle_{\mathrm{cl}}$ being just some classical (commutative) expectation value, which we take to have a generic form

$$\langle f(H) \rangle_{\mathrm{cl}} \equiv \int \mathrm{d}H \, \mathrm{e}^{-N \operatorname{Tr} V(H)} \, f(H), \tag{18}$$

where $V(H)$ is some (usually polynomial) potential.

This definition (16) again gives the rule of how to calculate mixed moments out of the separate moments (if the matrices are not centred, i.e. if $\langle H_i \rangle \neq 0$, we use the trick of renaming them as $\tilde{H}_i \equiv H_i - \langle H_i \rangle$); since by definition $\langle \tilde{H}_1 \tilde{H}_2 \rangle = 0$

$$\langle H_1 H_2 \rangle = \langle H_1 \rangle \langle H_2 \rangle, \tag{19}$$

as in (4). However, fourth moments read this time

$$\langle H_1 H_2 H_1 H_2 \rangle = \langle H_1^2 \rangle \langle H_2 \rangle^2 + \langle H_1 \rangle^2 \langle H_2^2 \rangle - \langle H_1 \rangle^2 \langle H_2 \rangle^2, \tag{20}$$

$$\langle H_1 H_1 H_2 H_2 \rangle = \langle H_1^2 \rangle \langle H_2^2 \rangle. \tag{21}$$

Note that the above two moments are different, since now the variables $H_1$ and $H_2$ are non-commutative. So freeness is in some sense a much more restrictive property than well-known independence; mixed moments are the combinations of products of the individual moments. It turns out that it is precisely freeness that extends all the important features of independence to

the non-commutative case; non-commutative probability together with the notion of freeness is known under the name of *free random variables* FRV *calculus*.

We are now aiming at an addition law for free Hermitian random matrices, just as we have done in classical probability. The fundamental result in FRV calculus [15] is that one can introduce (via the so-called *non-crossing partitions* [16]) an analogue of cumulants, the so-called *free cumulants*, which again obey the *linearity condition*,

$$k_{H_1+H_2,n} = k_{H_1,n} + k_{H_2,n}. \tag{22}$$

The point is that we cannot simply relate the moments of $H_1 + H_2$ to the moments of $H_1$ and $H_2$ separately, as is done in the commutative case, since now mixed moments of centred variables do not factorize; nevertheless, it is possible to construct other objects, the free cumulants, that have this property and thus lead to the addition law, but their relation to the moments is far more involved than before.

This was the essence of the construction. When it is established, what remains is to construct direct translations of notions from classical probability to FRV. First, we make a generating function of the moments, like we did for introducing a characteristic function (7). Here it is convenient to choose such a function of the form

$$G_H(z) \equiv \sum_{n\geqslant 0} \frac{\langle H^n \rangle}{z^{n+1}} = \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{z1_N - H} \right\rangle_{\mathrm{cl}}, \tag{23}$$

which is called the *Green's function*. Similarly, as we gathered all the cumulants in the logarithm of the characteristic function in (11), we can encode all the free cumulants in terms of a single function called Voiculescu's *R-transform*:

$$R_H(z) \equiv \sum_{n\geqslant 0} k_{H,n+1} z^n. \tag{24}$$

(We do not need to put factorial in the denominator as variables do not commute. The shift in $n$ is introduced for further convenience.) Now we can finally write the FRV *addition law*:

$$R_{H_1+H_2}(z) = R_{H_1}(z) + R_{H_2}(z). \tag{25}$$

It is very elegant, but what should worry the reader is that we have mentioned that the relation between the free cumulants and the moments is very entangled, which might mean that it is a very tough task to derive $R_H(z)$, i.e. the generating function of the free cumulants, from $G_H(z)$, i.e. the generating function of the moments. (On the classical level it is just a matter of taking logarithm.) Surprisingly enough, there exists a simple resolution of this problem, namely if we slightly redefine the $R$-transform,

$$B_H(z) \equiv R_H(z) + \frac{1}{z}, \tag{26}$$

we get the fundamental relation

$$B_H(G_H(z)) = G_H(B_H(z)) = z, \tag{27}$$

i.e. the function $B(z) = R(z) + 1/z$ is the *functional inverse* of the Green's function. We call it Blue's function, after Tony Zee who proposed this name and popularized FRV among the physicists [18]. Once we have the Green's function, i.e. we know the moments, it is straightforward to get Blue's function, i.e. the free cumulants, which are additive. Let us also rewrite the addition law (25) in the language of Blue's function,

$$B_{H_1+H_2}(z) = B_{H_1}(z) + B_{H_2}(z) - \frac{1}{z}. \tag{28}$$

So the algorithm of adding two free Hermitian random matrices may be summarized as follows:

- Assuming that we can construct the Green's functions $G_{H_1}(z)$ and $G_{H_2}(z)$, we obtain Blue's functions $B_{H_1}(z)$ and $B_{H_2}(z)$ by the functional inversion (27).
- It amounts to apply the addition law (28) to get $B_{H_1+H_2}(z)$.
- Finally, we again invert it functionally to find $G_{H_1+H_2}(z)$.

Let us recall how to use the Green's function to obtain spectral properties of $H$. The fundamental problem in random matrix theory is to find the averaged distribution of the eigenvalues $\lambda_i$ of $H$,

$$\rho_H(\lambda) = \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle_{\mathrm{cl}}. \tag{29}$$

The Green's function can be easily used to reconstruct this density according to

$$\rho_H(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathrm{Im} G_H(\lambda + \mathrm{i}\epsilon), \tag{30}$$

which stems from the use of the well-known formula $1/(\lambda + \mathrm{i}\epsilon) = \mathrm{pv}1/\lambda - \mathrm{i}\pi\delta(\lambda)$. Hence the eigenvalues density can be read out of the discontinuities of the imaginary part of the Green's function. In particular, for the addition of two Hermitian matrices, the above addition algorithm constitutes at the level of densities a generalization of the convolution (10), called the *additive free convolution*,

$$\rho_{H_1+H_2} = \rho_{H_1} \boxplus \rho_{H_2}. \tag{31}$$

As a pedagogical example, let us consider the case of Gaussian randomness, which at the level of Hermitian random matrices is known as Gaussian unitary ensemble (GUE), defined by the potential (see (17), (18))

$$V(H) = H^2. \tag{32}$$

In the large-$N$ limit, the Green's function (23) is calculable analytically by the saddle-point method [2] with the result

$$G_H(z) = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right). \tag{33}$$

Its singularities form the single cut $[-2, 2]$ on the real line, and we recover from (30) the seminal *Wigner's semicircle law* [19],

$$\rho_H(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2}. \tag{34}$$

The Wigner's semicircle law constitutes a matrix analogue of the Gaussian distribution. Now let us pick two GUE matrices, $H_1$ and $H_2$, and try to add them according to the above FRV algorithm, which is an analogue of what we have done in subsection 2.1. We substitute $z \to B(z)$ and we use formula (27) to get easily the functional inversion of the Green's function,

$$B_{H_{1,2}}(z) = z + \frac{1}{z}, \tag{35}$$

then we exploit the FRV addition law (28) to obtain

$$B_{H_1+H_2}(z) = 2z + \frac{1}{z}, \tag{36}$$

and again invert it functionally,

$$G_{H_1+H_2}(z) = \frac{1}{4}\left(z - \sqrt{z^2 - 8}\right), \tag{37}$$

which leads to the eigenvalues density

$$\rho_{H_1+H_2}(\lambda) = \frac{1}{4\pi}\sqrt{8-\lambda^2}; \tag{38}$$

this is again the Wigner's semicircle law, but rescaled by the factor of $\sqrt{2}$, in analogy with the previous classical computation. (The result $R_{H_1+H_2}(z) = 2z$ also tells us that only the second free cumulant $k_{H_1+H_2,2} = 2$ is non-vanishing, which is an expected feature of the matrix analogue of the centred Gaussian distribution.)

The above example demonstrates that the techniques of FRV calculus offer a powerful shortcut when we search for the distribution of spectra coming from sums of ensembles. The addition law can be used regardless of the potential $V(H)$, e.g. for higher order polynomials, Lévy potentials, non-random ensembles, etc.

We have established the passage (1); let us now use what we have learned here to tackle the passage (2) on the non-Hermitian level.

## 3. Addition law for non-Hermitian random matrices

### *3.1. Introduction*

The crucial difference which arises in the non-Hermitian case (we denote an arbitrary non-Hermitian random matrix by $X$) is that the eigenvalues of $X$ are complex in general; in the large-$N$ limit they form two-dimensional domains ('islands') on the complex plane, in contrary to one-dimensional cuts in the Hermitian case. The Green's function loses its analyticity, i.e. it is analytic (holomorphic) only outside the eigenvalues' domains, whereas in the Hermitian case it is holomorphic everywhere except some one-dimensional cuts; hence the power series expansion no longer captures the full information about the Green's function, and it is exactly its non-holomorphic behaviour that determines the eigenvalues' distribution on the two-dimensional supports.

This phenomenon can be easily seen even in the simplest non-Hermitian ensemble, the Girko–Ginibre one [20, 21], with a complex random matrix $X$ and the Gaussian measure

$$V(X, X^\dagger) = XX^\dagger. \tag{39}$$

It is easy to verify that all the moments vanish, $\langle \mathrm{Tr}\, X^n\rangle = 0$ for $n > 0$ (note however that $\langle \mathrm{Tr}(XX^\dagger)^n\rangle \neq 0$), so expansion (23) would give $G_X(z) = \frac{1}{z}$ everywhere on the complex plane which is an incorrect result. The true answer is that $G_X(z) = 1/z$ is valid only for $|z| > 1$, whereas for $|z| < 1$ we have the non-holomorphic Green's function $G_X(z, \bar{z}) = \bar{z}$.

### *3.2. Electrostatic analogy and regularized Green's function*

We approach this problem by exploiting the analogy to two-dimensional electrostatics [22–26]. Let us define the *electrostatic potential*,

$$F_X(z, \bar{z}) \equiv \frac{1}{N}\langle \mathrm{Tr}\ln((z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N)\rangle_{\mathrm{cl}}, \tag{40}$$

and note that its Laplacian gives the spectrum of $X$ as a charge density

$$\frac{1}{\pi}\partial_z\partial_{\bar{z}}F_X(z, \bar{z}) = \rho_X(z, \bar{z}) \tag{41}$$

(this is the Poisson equation), due to the representation of the two-dimensional (complex) Dirac's delta,

$$\delta^{(2)}(z) = \frac{1}{\pi}\frac{\epsilon^2}{(|z|^2 + \epsilon^2)^2}. \tag{42}$$

We can rewrite (41) by introducing the electric field,

$$G_X(z,\bar{z}) \equiv \partial_z F_X(z,\bar{z}) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{\bar{z}1_N - X^\dagger}{(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N}\right\rangle_{\mathrm{cl}}, \tag{43}$$

and the connection with $\rho_X(z,\bar{z})$ is now the Gauss' law,

$$\rho_X(z,\bar{z}) = \frac{1}{\pi}\partial_{\bar{z}} G_X(z,\bar{z}). \tag{44}$$

We see that there has appeared an additional factor of $\bar{z}1_N - X^\dagger$ both in the numerator and denominator, and moreover, the denominator has been regularized by $\epsilon^2 1_N$. This *regularized Green's function* has now profoundly drawn us closer to the solution of our problem. First, this is because it is defined everywhere, also inside the eigenvalues' domains, so through it we have access to the interiors of the domains; however, there it is non-holomorphic. Second, it provides the eigenvalues' density via the simple formula (44), which is exactly what we need. Third, outside the domains we can safely set the regulator $\epsilon$ to zero, which makes the regularized Green's function equal to the standard one (23).

The shortcoming of this construction is due to the quadratic structure in the denominator in (43), in contrast to the linear structure in (23), which jeopardizes practical calculations. To resolve this obstacle, we take one more step in our construction.

### *3.3. Matrix-valued Green's function*

Instead of working with the complicated object (43), one can make the following trick [27]. Define the *matrix-valued Green's function* as the $2\times 2$ matrix

$$\mathcal{G}_X(z,\bar{z}) \equiv \frac{1}{N}\left\langle \mathrm{b\,Tr}\left(\begin{array}{c|c} z1_N - X & \mathrm{i}\epsilon 1_N \\ \hline \mathrm{i}\epsilon 1_N & \bar{z}1_N - X^\dagger \end{array}\right)^{-1}_{2N\times 2N}\right\rangle_{\mathrm{cl}}, \tag{45}$$

where the block-trace

$$\mathrm{b\,Tr}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)_{2N\times 2N} \equiv \left(\begin{array}{c|c} \mathrm{Tr}\,A & \mathrm{Tr}\,B \\ \hline \mathrm{Tr}\,C & \mathrm{Tr}\,D \end{array}\right)_{2\times 2}. \tag{46}$$

Explicitly,

$$\mathcal{G}_X(z,\bar{z}) = \left(\begin{array}{c|c} \mathcal{G}_X^{11}(z,\bar{z}) & \mathcal{G}_X^{1\bar{1}}(z,\bar{z}) \\ \hline \mathcal{G}_X^{11}(z,\bar{z}) & \mathcal{G}_X^{1\bar{1}}(z,\bar{z}) \end{array}\right)_{2\times 2}, \tag{47}$$

with

$$\mathcal{G}_X^{11}(z,\bar{z}) = \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{\bar{z}1_N - X^\dagger}{(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N}\right\rangle_{\mathrm{cl}} = G_X(z,\bar{z}), \tag{48}$$

$$\mathcal{G}_X^{\bar{1}1}(z,\bar{z}) = \mathcal{G}_X^{1\bar{1}}(z,\bar{z}) = \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{-\mathrm{i}\epsilon}{(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N}\right\rangle_{\mathrm{cl}}, \tag{49}$$

$$\mathcal{G}_X^{\bar{1}\bar{1}}(z,\bar{z}) = \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{z1_N - X}{(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N}\right\rangle_{\mathrm{cl}} = \overline{G_X(z,\bar{z})}. \tag{50}$$

In particular, we see that the upper left corner of the matrix-valued Green's function (45) precisely equals the regularized Green's function (43).

This block approach now resolves our previous problem of the quadratic denominator. Indeed, the matrix-valued Green's function (45) can be re-expressed in the form identical to that of the usual Green's function (23) but on the matrix level,

$$\mathcal{G}_X(z,\bar{z}) = \frac{1}{N}\left\langle \mathrm{b\,Tr}\,\frac{1}{Z_\epsilon \otimes 1_N - X^{\mathrm{D}}}\right\rangle_{\mathrm{cl}}, \tag{51}$$

where

$$Z_\epsilon \equiv \left(\begin{array}{c|c} z & \mathrm{i}\epsilon \\ \hline \mathrm{i}\epsilon & \bar{z} \end{array}\right)_{2\times 2} \tag{52}$$

is the matrix analogue of the variable $z$, and

$$X^{\mathrm{D}} \equiv \left(\begin{array}{c|c} X & \\ \hline & X^\dagger \end{array}\right)_{2N\times 2N} \tag{53}$$

is the 'duplicated' matrix $X$. The necessity of dealing with $2\times 2$ matrices instead of numbers, exchanging $z1_N$ with $Z_\epsilon \otimes 1_N$ and $X$ with $X^{\mathrm{D}}$ is the price we have to pay for linearizing the denominator. And indeed, now the denominator is linear in $X^{\mathrm{D}}$ (not in $X$), so the matrix-valued Green's function is completely determined by the knowledge of all the matrix-valued moments,

$$\langle \mathrm{b}\,\mathrm{Tr}\, Z_\epsilon^{-1} X^{\mathrm{D}} Z_\epsilon^{-1} X^{\mathrm{D}} \ldots \rangle_{\mathrm{cl}}, \tag{54}$$

again in analogy with the Hermitian case.

We summarize the general properties [27] of the matrix-valued Green's function. Each component carries important information about the stochastic properties of the system.

- The 1 1-element (the $\bar{1}\bar{1}$-element is just its complex conjugation) is the regularized Green's function. We will see that the equations that will be found for $\mathcal{G}_X(z,\bar{z})$ always admit two solutions for its 1 1-element, one of them being valid outside the eigenvalues' domains and equal to the standard Green's function $G_X(z)$ (called the *holomorphic solution*), and the other valid also inside the domains and equal to the regularized Green's function $G_X(z,\bar{z})$ (called the *non-holomorphic solution*). The second one leads via the Gauss' law (44) to the eigenvalues' density. The first one is not spurious, as one may now wonder, but it represents the generating function for the moments of $X$ [28], so although it cannot reproduce the spectrum of the eigenvalues of $X$ (which calls for the usage of the non-holomorphic sector), it still provides important information about $X$.
- It is curious that also the off-diagonal elements, $1\bar{1}$ and $\bar{1}1$, have an interesting interpretation [29]. Namely, their product,

$$C_X(z,\bar{z}) \equiv \mathcal{G}_X^{1\bar{1}}(z,\bar{z})\mathcal{G}_X^{\bar{1}1}(z,\bar{z}), \tag{55}$$

  represents the correlator between left and right eigenvectors of $X$, introduced in [30]

$$\frac{1}{N}\left\langle \sum_{i=1}^{N} (L_i|L_i)(R_i|R_i)\delta^{(2)}(z-\lambda_i)\right\rangle_{\mathrm{cl}} = -\frac{1}{\pi} C_X(z,\bar{z}). \tag{56}$$

  In particular, it allows us to find the shape of the bordering 'coastline' of the eigenvalues' 'islands', since on the borderline that correlator must vanish, which gives us the equation of the borderline of the eigenvalues' domains,

$$C_X(z,\bar{z}) = 0. \tag{57}$$

A similar approach, discussed at this workshop by Joshua Feinberg, under the name 'Hermitization method' [31–33], uses basically an alternative representation for the matrix structure of the regularized Green's function,

$$\tilde{\mathcal{G}}_X(z,\bar{z}) \equiv \frac{1}{N}\left\langle \mathrm{b}\,\mathrm{Tr}\left(\begin{array}{c|c} 1_N & z1_N - X \\ \hline \bar{z}1_N - X^\dagger & -\epsilon^2 1_N \end{array}\right)^{-1}_{2N\times 2N}\right\rangle_{\mathrm{cl}}, \tag{58}$$

which is Hermitian and admits to use well-established Hermitian methods; both versions lead to similar results.

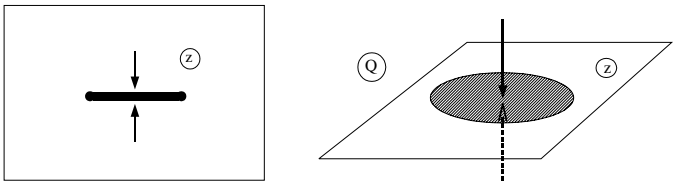


**Figure 1.** (*a*) Hermitian case (left), real spectral function obtained from approaching cuts on a complex plane; (*b*) Non-Hermitian case (right), complex spectral function from approaching the non-analytical region from 'orthogonal' directions in quaternion space.

The presented construction has allowed considerable progress in analysing non-Hermitian ensembles. In particular, it has led to the generalization of the concept of the Blue's function for non-Hermitian random matrices, as first proposed in [34] and confirmed in [31, 32]. This extension has been defined as a matrix-valued function of a matrix-valued variable that satisfies

$$\mathcal{B}_X(\mathcal{G}_X(z,\bar{z})) = Z_\epsilon. \tag{59}$$

However, it was not entirely clear how to construct explicitly this function in full generality, and often some additional insight, coming from diagrammatic methods was necessary. In the next subsection we will circumvent this obstruction by introducing the so-called *quaternion approach* [1].

### *3.4. Quaternion Green's and Blue's functions*

In the Hermitian case, working with the *complex* Green's function allowed us to infer *real* spectral distributions from the discontinuities on the complex plane, as visualized in figure 1(*a*). It is tempting to find a similar method in the non-Hermitian case of *complex* spectra. A natural generalization is the algebra of *quaternions*; even though such speculations appeared in the literature [32], an explicit realization appeared only very recently [1].

Schematically, such a scenario is visualized in figure 1(*b*); to find the complex eigenvalues' distribution, we approach two sides of complex eigenvalues domain from 'orthogonal' (in the quaternion space) directions to the complex plane. Actually, a specific realization can be naturally achieved by a straightforward generalization of the matrix-valued Green's function (43); from now on let us call the *quaternion Green's function* the object

$$\mathcal{G}_X(Q) \equiv \frac{1}{N}\left\langle \mathrm{b\,Tr}\,\frac{1}{Q\otimes 1_N - X^{\mathrm{D}}}\right\rangle_{\mathrm{cl}}, \tag{60}$$

where $Q$ is an *arbitrary quaternion*,

$$Q = \left(\begin{array}{c|c} a & \mathrm{i}\bar{b} \\ \hline \mathrm{i}b & \bar{a} \end{array}\right)_{2\times 2} = x_0 1_2 + \mathrm{i}\vec{x}\cdot\vec{\sigma} \tag{61}$$

(here $\sigma_i$ are the usual Pauli matrices and $a = x_0 + \mathrm{i}x_3$, $b = x_1 + \mathrm{i}x_2$). This defines $\mathcal{G}_X$ as a quaternion function of a quaternion variable.

The difference to the original meaning of this notion is that we have replaced $Z_\epsilon$ by a general quaternion $Q$ thus promoting $\mathcal{G}_X$ to be a *function of* $Q$. Particularly, for $Q = Z_\epsilon$ we arrive at the former meaning. The basic idea of the quaternion extension is that $\mathcal{G}_X$ has non-trivial properties as a function of $Q$, which cannot be seen when restricted to the case of $Q = Z_\epsilon$. In particular, the FRV calculus becomes simpler in the non-Hermitian case, since to invert functionally the quaternion Green's function, i.e. to find the *quaternion Blue's function*,

$$\mathcal{G}_X(\mathcal{B}_X(Q)) = \mathcal{B}_X(\mathcal{G}_X(Q)) = Q. \tag{62}$$

For *any* quaternion $Q$, we can utilize algebraic properties of the quaternions. Accordingly, the quaternion addition law holds [1],

$$\mathcal{B}_{X_1+X_2}(Q) = \mathcal{B}_{X_1}(Q) + \mathcal{B}_{X_2}(Q) - \frac{1}{Q}, \tag{63}$$

for any $Q$ and two non-Hermitian free random matrices $X_1$ and $X_2$.

### *3.5. Practical use of quaternion Green's and Blue's functions*

Having established the foundations of the quaternion approach, we can ask the question of their practical advantages. Indeed, it is obvious that for example to pass from $H_1$, $H_2$ to $H_1 + \mathrm{i}H_2$, see (2), we will need not only the quaternion addition law (63), but also a tractable form of the quaternion Blue's function of a Hermitian random matrix $H$, as well as a prescription to compute this function for $\mathrm{i}H$ provided it is known for $H$.

Both these problems have been solved [1] and we just present final results. So the quaternion Green's function for an arbitrary Hermitian random matrix $H$ reads

$$\mathcal{G}_H(Q) = \gamma_H(q, \bar{q})1_2 - \gamma'_H(q, \bar{q})Q^\dagger, \tag{64}$$

where $q = x_0 + \mathrm{i}|\vec{x}|$ and $\bar{q} = x_0 - \mathrm{i}|\vec{x}|$ are two conjugated eigenvalues of the quaternion $Q$ (61), whereas $\gamma_H$ and $\gamma'_H$ are two scalar functions depending only on $q$ and $\bar{q}$, and given explicitly by

$$\gamma_H(q, \bar{q}) \equiv \frac{qG_H(q) - \bar{q}G_H(\bar{q})}{q - \bar{q}}, \tag{65}$$

$$\gamma'_H(q, \bar{q}) \equiv \frac{G_H(q) - G_H(\bar{q})}{q - \bar{q}}. \tag{66}$$

This formula can be inverted functionally, which leads to a symmetrical result for the quaternion Blue's function of $H$,

$$\mathcal{B}_H(Q) = \beta_H(q, \bar{q})1_2 - \beta'_H(q, \bar{q})Q^\dagger, \tag{67}$$

where $\beta_H$ and $\beta'_H$ are similar to $\gamma_H$ and $\gamma'_H$ but with $G_H$ in the place of $B_H$:

$$\beta_H(q, \bar{q}) \equiv \frac{qB_H(q) - \bar{q}B_H(\bar{q})}{q - \bar{q}}, \tag{68}$$

$$\beta'_H(q, \bar{q}) \equiv \frac{B_H(q) - B_H(\bar{q})}{q - \bar{q}}. \tag{69}$$

The second problem can be solved as well and gives the scaling properties of the quaternion Green's and Blue's functions,

$$\mathcal{G}_{gX}(Q) = \mathcal{G}_X\left(\left(\begin{array}{c|c} \frac{1}{g} & \\ \hline & \frac{1}{\bar{g}} \end{array}\right) Q\right)\left(\begin{array}{c|c} \frac{1}{g} & \\ \hline & \frac{1}{\bar{g}} \end{array}\right), \qquad g \in \mathbb{C}\backslash\{0\}, \tag{70}$$

and

$$\mathcal{B}_{gX}(Q) = \left(\begin{array}{c|c} g & \\ \hline & \bar{g} \end{array}\right)\mathcal{B}_X\left(Q\left(\begin{array}{c|c} g & \\ \hline & \bar{g} \end{array}\right)\right), \qquad g \in \mathbb{C}\backslash\{0\}. \tag{71}$$

These two relations generalize the well-known scaling properties of standard (in Hermitian RMT) Green's and Blue's functions,

$$G_{gX}(z) = \frac{1}{g}G_X\left(\frac{1}{g}z\right), \qquad g \in \mathbb{C}\backslash\{0\}, \tag{72}$$

and

$$B_{gX}(z) = gB_X(gz), \qquad g \in \mathbb{C}\backslash\{0\}. \tag{73}$$

Now the non-Hermitian problem is to large extent reduced to Hermitian random matrix theory: knowing the Hermitian inputs of our problem, i.e. the standard Green's functions (23) $G_{H_1}(z)$ and $G_{H_2}(z)$, we can derive the full quaternion Green's function (60), using the quaternion addition law (63) and the explicit formulae (64), (67), (70) and (71), so

$$G_{H_1}(z), G_{H_2}(z) \longrightarrow \mathcal{G}_{H_1+\mathrm{i}H_2}(Q). \tag{74}$$

The algorithm is just as follows:

- Given are $G_{H_1}(z)$ and $G_{H_2}(z)$. This is the Hermitian RMT input.
- Compute $\mathcal{B}_{H_1}(Q)$ and $\mathcal{B}_{H_2}(Q)$ using (67).
- Compute $\mathcal{B}_{iH_2}(Q)$ from $\mathcal{B}_{H_2}(Q)$ and (71) with $g = \mathrm{i}$.
- Use the quaternion addition law (63) to get $\mathcal{B}_{H_1+\mathrm{i}H_2}(Q)$.
- Invert it functionally according to (62) to obtain $\mathcal{G}_{H_1+\mathrm{i}H_2}(Q)$. Actually, it is not necessary to invert it for all $Q$, it is sufficient to do the inversion at this moment for $Q = Z = \mathrm{diag}(z, \bar{z})$, as the upper left corner of this $2 \times 2$ matrix is the desired regularized Green's function (43). Finally, one recovers the two-dimensional eigenvalues' density $\rho_{H_1+\mathrm{i}H_2}(z, \bar{z})$ from (44).

This algorithm is plausible, but one can worry how to perform in general a functional inversion of a rather complicated quaternion function of a quaternion variable. This procedure was reduced to even simpler algebraic structure [1]. We quote here the result, pointing at the simplifications achieved.

- Write two equations

$$B_{H_1}(g) = x + \frac{m}{g}, \tag{75}$$

$$B_{H_2}(g') = y + \frac{1-m}{g'}, \tag{76}$$

where $z \equiv x + \mathrm{i}y$, with three unknown quantities, complex $g, g'$ and real $m$. Express $g$ and $g'$ via $m$.

- Compute $m$ from the third equation,

$$|g| = |g'|. \tag{77}$$

- Derive $g + \bar{g}$, $g' + \overline{g'}$ and $|g|^2$ from the above two steps.
- The non-holomorphic Green's function and the correlator between left and right eigenvectors for $H_1 + \mathrm{i}H_2$ are given by

$$G_{H_1+\mathrm{i}H_2}(x, y) = \frac{g+\bar{g}}{2} - \mathrm{i}\frac{g'+\overline{g'}}{2}, \tag{78}$$

$$C_{H_1+\mathrm{i}H_2}(x, y) = \left(\frac{g+\bar{g}}{2}\right)^2 + \left(\frac{g'+\overline{g'}}{2}\right)^2 - |g|^2. \tag{79}$$

The borderline's equation is

$$\left(\frac{g+\bar{g}}{2}\right)^2 + \left(\frac{g'+\overline{g'}}{2}\right)^2 = |g|^2. \tag{80}$$

Quaternion approach is quite useful from operational point of view, at least in the case of $H_1 + \mathrm{i}H_2$; it also works successfully in other cases, e.g. in the case of models involving unitary matrices [35].

*3.6. Example: Girko–Ginibre model*

As a pedagogical example let us come back to the Girko–Ginibre model mentioned in subsection 3.1, which can be equally well reformulated as an addition $H_1 + \mathrm{i}H_2$, where both $H_{1,2}$ are free GUE matrices (subsection 2.2).

Let us first construct the quaternion functions explicitly from the algorithm, without referring to its simplified version. We have (35)

$$B_{H_{1,2}}(z) = z + \frac{1}{z}, \tag{81}$$

hence from (67) we get

$$\mathcal{B}_{H_{1,2}}(Q) = Q + \frac{1}{Q}. \tag{82}$$

Next, we use (71) and obtain

$$\mathcal{B}_{\mathrm{i}H_2}(Q) = \mathrm{i}\sigma_3 Q \mathrm{i}\sigma_3 + \frac{1}{Q}. \tag{83}$$

The quaternion addition law (63) thus states that

$$\mathcal{B}_{H_1+\mathrm{i}H_2}(Q) = Q + \mathrm{i}\sigma_3 Q \mathrm{i}\sigma_3 + \frac{1}{Q}. \tag{84}$$

Here we substitute $Q \to \mathcal{G}_{H_1+\mathrm{i}H_2}(z, \bar{z})$, which gives the $2 \times 2$ matrix equation

$$\left(\begin{array}{c|c} z & 0 \\ \hline 0 & \bar{z} \end{array}\right) = \left(\begin{array}{c|c} \mathcal{G}_X^{11} & \mathcal{G}_X^{1\bar{1}} \\ \hline \mathcal{G}_X^{11} & \mathcal{G}_X^{1\bar{1}} \end{array}\right) + \frac{1}{\operatorname{Det}\mathcal{G}_X}\left(\begin{array}{c|c} \mathcal{G}_X^{\bar{1}\bar{1}} & -\mathcal{G}_X^{1\bar{1}} \\ \hline -\mathcal{G}_X^{11} & \mathcal{G}_X^{11} \end{array}\right) - \left(\begin{array}{c|c} \mathcal{G}_X^{11} & -\mathcal{G}_X^{1\bar{1}} \\ \hline -\mathcal{G}_X^{11} & \mathcal{G}_X^{1\bar{1}} \end{array}\right). \tag{85}$$

The equation in the upper right corner thus reads

$$\mathcal{G}_X^{1\bar{1}}\left(2 - \frac{1}{\operatorname{Det}\mathcal{G}_X}\right) = 0, \tag{86}$$

which means either

$$\mathcal{G}_X^{1\bar{1}} = 0, \qquad \text{then} \qquad \mathcal{G}_X^{11} = \frac{1}{z}, \tag{87}$$

or

$$\operatorname{Det}\mathcal{G}_X = \frac{1}{2}, \qquad \text{then} \qquad \mathcal{G}_X^{11} = \frac{\bar{z}}{2}. \tag{88}$$

We explicitly see two kinds of solutions mentioned above in subsection 3.3, the holomorphic one and the non-holomorphic one. The matching condition (the solutions match on the borderline of the islands of eigenvalues),

$$\frac{1}{z} = \frac{\bar{z}}{2}, \qquad \text{i.e.} \quad x^2 + y^2 = 2, \tag{89}$$

gives the borderline of the domains, which here is the circle of radius $\sqrt{2}$. (We got here the radius $\sqrt{2}$, since we add two Hermitian ensembles, whereas the original Ginibre case corresponds to adding to symmetric ensembles). Inside it, the eigenvalues' density can be found from (44) as

$$\rho_{H_1+\mathrm{i}H_2}(z, \bar{z}) = \frac{1}{\pi}\partial_{\bar{z}}\frac{\bar{z}}{2} = \frac{1}{2\pi}, \tag{90}$$

i.e. the complex eigenvalues fill the circle $x^2 + y^2 \leqslant 2$ with the uniform density $1/2\pi$.

The same result can be obtained from the simplified version of the algorithm mentioned at the end of the previous subsection. Here let us use for a slightly more general model

$H_1 + \mathrm{i}cH_2$, with some coupling constant $c \in \mathbb{R}$, to be used in the next section. The equations (75), (76) are quadratic, $g^2 - xg + 1 - m = 0$, $c^2 g'^2 - yg' + m = 0$, so Vieta's rules yield $g + \bar{g} = x$, $|g|^2 = 1 - m$, $g' + \overline{g'} = \frac{y}{c^2}$, $|g'|^2 = \frac{m}{c^2}$, so (77) is $1 - m = \frac{m}{c^2}$, hence $m = \frac{c^2}{1+c^2}$, which finally leads to $G_{H_1+\mathrm{i}cH_2}(x, y) = \frac{1}{2}\left(x - \mathrm{i}\frac{y}{c^2}\right)$ and $C_{H_1+\mathrm{i}cH_2}(x, y) = \frac{1}{4}\left(x^2 + \frac{y^2}{c^4}\right) - \frac{1}{1+c^2}$. In particular, the borderline's equation reads $x^2 + \frac{y^2}{c^4} = \frac{4}{1+c^2}$, which is an ellipse, reducing for $c = 1$ to a circle. Here the use of the equations (75)–(80) is only a minor advantage; however for more involved models this procedure simplifies practical calculations considerably.

## 4. Conformal mapping

Finally, let us present another astonishing link [17] between the Hermitian model $H_1 + H_2$ and its non-Hermitian counterpart $H_1 + \mathrm{i}H_2$, based on the FRV formalism. The existence of the holomorphic and non-holomorphic solutions provides a powerful way to evaluate the domains of eigenvalues of $H_1 + \mathrm{i}H_2$; namely, the borderlines of these supports can be derived in a general way using a *conformal transformation* that maps the cuts of the Hermitian ensemble onto the boundaries of its non-Hermitian analogue. The idea is to use the standard (Hermitian RMT based) addition law (28) to the *non-Hermitian* case, but restricted to the domains of analyticity of both ensembles. Let us just consider the case of arbitrary $H_1 \equiv M$ and $H_2 \equiv cH \equiv c\mathrm{GUE}$, $c \in \mathbb{R}$ being a coupling constant. The addition law (28) for $M + H$, together with (35) and (73), says that

$$B_{M+cH}(u) = c^2 u + B_M(u). \tag{91}$$

Now we use the same addition law for $M + \mathrm{i}cH$, using the scaling property (73),

$$B_{M+\mathrm{i}cH}(u) = -c^2 u + B_M(u). \tag{92}$$

(Note that this is a consequence of the group property for $R$-transforms, $R_H(u) + R_{iH}(u) = 0$, i.e. the anti-Hermitian GUE nullifies the Hermitian GUE in the holomorphic domain.) These two equations yield

$$B_{M+\mathrm{i}cH}(u) = B_{M+cH}(u) - 2c^2 u. \tag{93}$$

Now let us substitute here $u \to G_{M+H}(z)$. Then

$$B_{M+\mathrm{i}cH}(G_{M+cH}(z)) = z - 2c^2 G_{M+cH}(z). \tag{94}$$

Let $w$ be the image of $z$ such that $G_{M+\mathrm{i}cH}(w) = G_{M+cH}(z)$. Then

$$w = z - 2c^2 G_{M+cH}(z). \tag{95}$$

This equation provides a conformal transformation mapping the holomorphic domain of the Hermitian ensemble $M + cH$, i.e. the complex plane except the cuts, onto the holomorphic domain of the non-Hermitian ensemble $M + \mathrm{i}cH$, i.e. the complex plane minus the domains, giving thus an efficient alternative method of dealing with the borderline of the domain of $M + \mathrm{i}cH$.

As an example, let us take $M = \mathrm{GUE}$ too. We thus have

$$B_{M+cH}(z) = (c^2 + 1)z + \frac{1}{z}, \qquad \text{i.e.} \quad G_{M+cH}(z) = \frac{z - \sqrt{z^2 - 4(c^2+1)}}{2(c^2+1)}, \tag{96}$$

so in particular the eigenvalues fill the cut $[-2\sqrt{c^2+1}, 2\sqrt{c^2+1}]$. This cut can be mapped to the borderline of the eigenvalues' domain of $M + \mathrm{i}cH$ (which we want to find via (95)). We

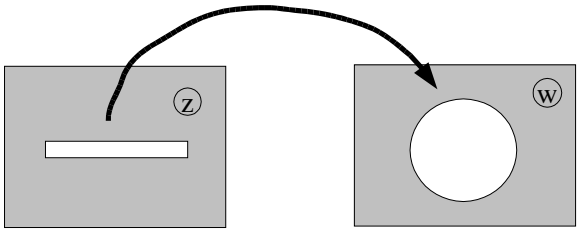


**Figure 2.** Conformal mappings for the case of Ginibre ensemble. The shaded regions represent the holomorphic domains.

already know from the end of the previous subsection that it is the ellipse $x^2 + \frac{y^2}{c^4} = \frac{4}{1+c^2}$, but we want to recover this in another way. The conformal map reads

$$w = \frac{1}{c^2+1}\left(z + c^2\sqrt{z^2 - 4(c^2+1)}\right). \tag{97}$$

We see that for $u = t \pm \mathrm{i}\epsilon$ and $t \in [-2\sqrt{c^2+1}, 2\sqrt{c^2+1}]$ the image $w(z)$ is exactly the ellipse $x^2 + \frac{y^2}{c^4} = \frac{4}{1+c^2}$. (See figure 2, where coupling $c$ is equal to one.)

Therefore, exploiting only the analytical properties of the holomorphic solution, we are able to infer the shape of the domain of eigenvalues. Note that this method cannot provide the density of eigenvalues inside the domain, since it is solely based on holomorphic properties. On the other hand, this method allows us to derive the moments of the distribution, since holomorphic and non-holomorphic Green's functions match at the boundary. In our example (with $c = 1$), the holomorphic Green's function reads $G_{M+H}(z) = 1/z$, hence all the moments vanish; from the exact solution we know that indeed this is the case, the distribution is uniform, so all the moments vanish. The opposite reasoning is not correct; for the two-dimensional domain we cannot infer the spectral distribution from the knowledge of all the moments.

## 5. Conclusions and prospects

In this paper we have pointed links between two very different systems: the random Hermitian system viewed as a 'sum' $H = H_1 + H_2$ and the random non-Hermitian system viewed as a 'sum' $X = H_1 + \mathrm{i}H_2$. When one thinks about $H_1$ and $H_2$ as Hamiltonians, the existence of any relations between the spectra of $H$ and $X$ is quite surprising. The connections presented here have emerged from the formalism of free random variables. Despite the foundations of the FRV were laid 15 years ago, this rich mathematical structure is not yet fully understood. We have exploited in this paper the analogy between classical probability theory and FRV, viewed as a matrix-valued, therefore non-commutative counterpart of the classical probability theory. It is tempting to ask whether FRV variables can also generate evolution in some external parameter ('time'), alike probabilistic distributions generate stochastic differential equations, paving therefore the road to 'free' statistical physics or 'free' quantum mechanics. These fundamental questions concerning dynamics of FRV are currently being addressed by mathematicians [36].

Another open problem is the connection between FRV and PT Hamiltonians. It is encouraging that certain class of PT Hamiltonians, corresponding to so-called Hatano–Nelson localization [10], was attempted to be formulated in the FRV formalism [12, 13]. We hope that this paper will trigger further studies of this problem.

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